Received: *April 12, 2000*, Accepted: *May 22, 2000*

Renormalization of quantum field theories on non-commutative $\mathbb{R}^d$, 1. Scalars

Iouri Chepelev and Radu Roiban

C.N. Yang Institute for Theoretical Physics, SUNY at Stony Brook
NY11794-3840, USA
Email: chepelev@insti.physics.sunysb.edu,
roiban@insti.physics.sunysb.edu

Abstract: A non-commutative Feynman graph is a ribbon graph and can be drawn on a genus g 2-surface with a boundary. We formulate a general convergence theorem for the non-commutative Feynman graphs in topological terms and prove it for some classes of diagrams in the scalar field theories. We propose a non-commutative analog of Bogoliubov-Parasiuk's recursive subtraction formula and show that the subtracted graphs from a class Ω_d satisfy the conditions of the convergence theorem. For a generic scalar non-commutative quantum field theory on $\mathbb{R}^d$, the class Ω_d is smaller than the class of all diagrams in the theory. This leaves open the question of perturbative renormalizability of non-commutative field theories. We comment on how the supersymmetry can improve the situation and suggest that a non-commutative analog of Wess-Zumino model is renormalizable.

Keywords: Bosonic Strings, Quantum Groups.

JHEP05(2000)037

Contents

1. Introduction

1.1 Historical background

What would physics be like if the space in which it took place was not a set of points, but a non-commutative space?[1] This was the question asked by Connes in ref. [2] where it was shown that a small modification of the usual picture of space-time gives an alternative explanation of the Higgs fields and of the way they appear in the Weinberg-Salam model.[2] Field theories on non-commutative spaces (NFT) [5, 6] are also interesting as a first step towards a formulation of quantum gravity which avoids standard problems [4].

[1] For a comprehensive account of non-commutative geometry, see ref. [1].
[2] See also [3].

JHEP05(2000)037

NFT became popular in the community of string theorists with the appearance of a paper by Connes, Douglas and Schwarz [7], where it was argued that M-theory in a constant three-form tensor background is equivalent to a super Yang-Mills theory on a non-commutative torus. For a review of developments following ref. [7], see ref. [8]. A second wave of interest towards NFT came with the work of Seiberg and Witten [9] which summarized and extended earlier ideas about the appearance of non-commutative geometry in string theory with a non-zero B-field.[3]

As stressed in ref. [8], the most pressing question regarding NFT is whether or not the quantum theory (NQFT) is well defined. The algebra of functions on the non-commutative $\mathbb{R}^d$ is isomorphic to the algebra of functions on commutative $\mathbb{R}^d$ with the multiplication of functions given by the $\star$-product

$$(\phi_1 \star \phi_2)(x) = e^{i\theta_{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\zeta^\nu}}\phi_1(x+\xi)\,\phi_2(x+\zeta)\Big|_{\xi=\zeta=0}\,. \tag{1.1}$$

The NFT action is the usual field theory action where the point-wise multiplication of the fields is replaced by the $\star$-product. The non locality of the NFT action in the position space looks bad at first sight and one might be led to conclude that NQFT is perturbatively non renormalizable. It was pointed out in ref. [8] that after deriving the Feynman rules for NQFT and studying the one-loop amplitude in momentum space one sees that the situation is actually rather good because the non-local interaction terms in the action provide oscillatory factors in the Feynman integrals. Indeed, one-loop renormalizability of non-commutative Yang-Mills (NYM) theory has been demonstrated in ref. [10]. In ref. [11] a non-commutative version of Wilson's lattice gauge theory formalism was developed. Such a formalism has the potential of clarifying issues of renormalization.

In ref. [5], Filk analyzed the structure of Feynman diagrams for the NQFT. He pointed out that the planar diagrams do not have oscillatory factors (involving loop momenta) coming from the non-local interaction terms, and thus the corresponding integrals are the same as in usual QFT.

The NQFT and QFT amplitudes for a planar graph G are related as

$$I_{\mathrm{NQFT}}(G,k) = e^{i\varphi(k)} I_{\mathrm{QFT}}(G,k)\,, \tag{1.2}$$

where k denotes the external momenta and $\varphi(k)$ is a phase depending only on k.

This means that the planar diagrams of NQFT diverge in the same way as the corresponding QFT diagrams. On the other hand, all non-planar diagrams have the oscillatory factors involving loop momenta. In ref. [12], Bigatti and Susskind claimed that the oscillatory factors would regulate divergent diagrams and make them finite, unless the diagrams contained divergent planar subdiagrams.[4]

[3] An extensive list of references on the subject can be found in ref. [9].

[4] The claim made in ref. [12] was partially supported by the supergravity calculations of gauge-invariant quantities of large-N non-commutative SYM in ref. [14]. See also ref. [13].

JHEP05(2000)037

1.2 Logic and structure of the paper

It is a desirable property of a diagram that it diverges only when it contains divergent planar subgraphs. The reason is the following. Let G be a non-planar NQFT graph which does not contain divergent planar subgraphs. If G were divergent, it would have to diverge *properly*, i.e. it should be possible to subtract the divergences by the introduction of counterterms which have the same form as those already occurring in the action. It is very unlikely that the divergent part of an integral involving oscillatory functions is proportional to the phase factors appearing in the lagrangian.[5]

In this paper we will analyze scalar field theories on non-commutative $\mathbb{R}^d$.[6] Our analysis consists of four steps:

1. A formulation of the convergence theorem for non-commutative Feynman graphs.
2. A recursion formula for the subtraction of divergences.
3. A proof that the application of the recursion formula to the integrand of a non-commutative graph yields an expression satisfying the conditions of the convergence theorem.
4. A proof that the subtraction procedure is equivalent to the introduction of the counterterms which have the same form as those already occurring in the action.

The step 1 (the convergence theorem) is central to the analysis. We will find precise conditions under which the claim made in ref. [12] regarding the convergence of non-commutative graphs is realized. A general NQFT Feynman graph with some external lines can be drawn on a genus g surface with a boundary (with one end of each external line being attached to the boundary). Let G be a NQFT Feynman graph. Draw it on a 2-surface Σ_g of genus g with a boundary $\partial\Sigma_g$. The non-trivial cycles of Σ_g are $a_1, b_1, \ldots, a_g, b_g$ (see figure 1). Cycles A, B, C and 0 are trivial.[7] Let γ be a subgraph of G. Let $c(\gamma)$ be the number of inequivalent non-trivial cycles of Σ_g spanned by the closed paths in γ. To the subgraph γ we assign an index $j(\gamma) = 0$ or 1 which characterizes the non planarity of γ with respect to the external lines of G.[8]

[5] In the minimal subtraction approach of ref. [10] there are some unusual divergent terms coming from the integrals involving periodic functions, but these terms cancel in the sum over all one-loop diagrams. The underlying reason for such cancellations seems to be the convergence of non-planar diagrams.

[6] Yang-Mills theory will be analyzed in ref. [15].

[7] A cycle on Σ_g is called nontrivial if it is a non-trivial element of the first homology group $H_1(\Sigma_g)$. In addition to the trivial cycles that are contractible to a point, there are trivial cycles which are not contractible to a point. For example, cycles A and C in figure 1 are trivial because $A = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ and $C = a_1 b_1 a_1^{-1} b_1^{-1}$, i.e. A and C are commutants. See ref. [16] for the details.

[8] For the precise definition of $j(\gamma)$ see section 3.

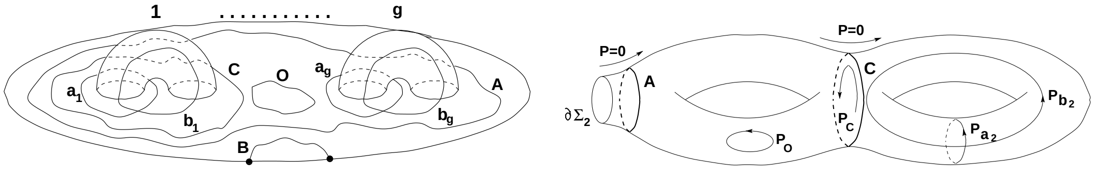
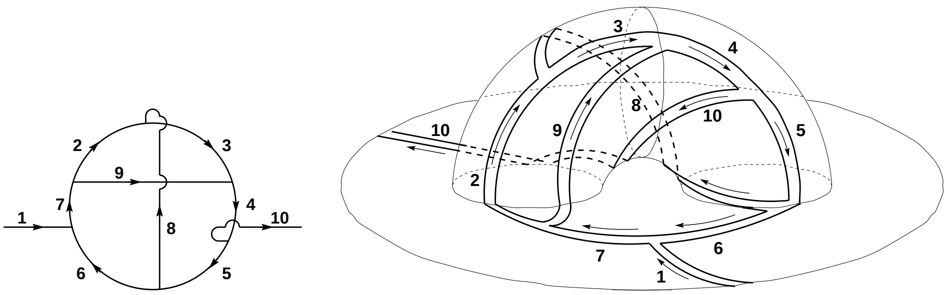


Figure 1: (a) A genus g surface with a boundary. (b) Momentum flow on a $g = 2$ surface Σ_2



(a) (b)

Figure 2: Illustration for example 1.

Example 1 A non-commutative Feynman graph in figure $2a$ is shown in figure $2b$ as a ribbon graph on a genus one 2-surface with a boundary. In d dimensions $\omega(\gamma)$, $c(\gamma)$ and $j(\gamma)$ for some subgraphs γ read as follows.

γ	$\omega(\gamma)$	$c(\gamma)$	$j(\gamma)$
(239)	$d-6$	0	0
(45679)	$d-10$	1	1
(2345678)	$2d-14$	2	1

Our convergence theorem can be stated as follows. *The 1PI graph G is convergent if and only if for any subgraph $\gamma \subseteq G$ at least one of the following conditions is satisfied:*

(1) $\omega(\gamma) - c(\gamma)d < 0$,

(2) $j(\gamma) = 1$.[9]

The meaning of this convergence theorem is the following. Each handle in figure 1 has two non-trivial cycles a_i and b_i. Let p_{a_i} and p_{b_i} be the total internal momenta flowing through the graph γ along the cycles a_i and b_i, respectively. The phase factor associated with each handle is $\exp(i\theta_{\mu\nu}\, p^{\mu}_{a_i}\, p^{\nu}_{b_i})$. As far as the convergence property of the graph is concerned, the effect of this phase factor is equivalent to reducing the

[9] For the scalar field theories discussed in this paper, $\omega(\gamma) = dL(\gamma) - 2I(\gamma)$, where L and I are the number of independent loops and internal lines of γ, respectively. It is assumed that the external momenta of the graph G are non exceptional.

number of loops by two. The condition $j(\gamma) = 1$ for a subgraph $\gamma \subset G$ means that a certain combination $\sum P_v$ of external momenta of G flows through γ and the path of the flow is *not homologous* to cycle B. The phase factor associated with such a flow is, schematically, $\exp(i\theta_{\mu\nu}(\sum P_v)^\mu q^\nu)$, where q is the loop momentum along a combination of cycles $a_1, \ldots, b_g$. This phase factor makes γ finite for arbitrary $\omega(\gamma)$ because of the exponential suppression at large external momenta (see section 3 for details).

The steps 2, 3 and 4 of our analysis are straightforward generalizations of the corresponding steps in the proof of renormalizability of commutative QFT. The only complication that arises due to the noncommutativity of the space is the distinction between topologically trivial and non-trivial subgraphs. For a given NQFT in d dimensions, we will show that the Feynman integral for any graph in a class Ω_d (to be defined in section 4) can be made finite by the application of the recursive subtraction formula to the integrand of that integral.

The paper is organized as follows. In section 2 we review Feynman rules for scalar NQFT and derive parametric integral representation for the amplitudes. We begin section 3 by analyzing the convergence properties of some simple diagrams and demonstrating the convergence of some classes of diagrams. We formulate a general convergence theorem, and show how it *explains*, in a unified manner, the convergence of the diagrams analyzed earlier. In section 4 we write down an analog of Bogoliubov-Parasiuk's recursion formula for the subtraction of divergences in NQFT and prove the convergence of subtracted integrals for the graphs from class Ω_d. We then suggest that the supersymmetric extension of scalar NQFT is renormalizable.

2. Scalar NQFT on the non-commutative $\mathbb{R}^d$

2.1 Definition of NQFT and Feynman rules

The non-commutative $\mathbb{R}^d$ is defined as follows. The coordinates $x^\mu (\mu = 1, \ldots, d)$ of commutative $\mathbb{R}^d$ are replaced by the self-adjoint operators $\hat{x}^\mu$ in a Hilbert space $\mathcal{H}$ satisfying the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = 2\, i\, \theta_{\mu\nu}\,, \qquad [\theta_{\mu\nu}, \hat{x}^\rho] = 0\,, \tag{2.1}$$

where θ is a non-degenerate $d \times d$ skew-symmetric matrix (d is even).

With a function $\phi(x)$ on the commutative space $\mathbb{R}^d$ one associates the operator $\Phi(\hat{x})$ acting in the Hilbert space $\mathcal{H}$ using the rule:

$$\Phi(\hat{x}) = \frac{1}{(2\pi)^d} \int d^d x\, d^d k\, e^{ik_\mu(\hat{x}^\mu - x^\mu)} \phi(x)\,. \tag{2.2}$$

Given an operator $\Phi(\hat{x})$, the function $\phi(x)$ can be obtained using

$$\phi(x) = \frac{1}{(2\pi)^{d/2}} \int d^d k\, e^{ik_\mu x^\mu}\, \mathrm{tr}\, \Phi(\hat{x})\, e^{-ik_\mu \hat{x}^\mu}\,, \tag{2.3}$$

where the trace tr is over the Hilbert space $\mathcal{H}$.

JHEP05(2000)037

To the product of two operators Φ_1 and Φ_2 corresponds a $\star$-product

$$(\phi_1 \star \phi_2)(x) = \frac{1}{(2\pi)^{d/2}} \int d^d k \, e^{ik_\mu x^\mu} \, \mathrm{tr}\left[\Phi_1 \Phi_2 e^{-ik_\mu \hat{x}^\mu}\right] = \\ = e^{i\theta_{\mu\nu} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \zeta^\nu}} \phi_1(x+\xi)\, \phi_2(x+\zeta)\Big|_{\xi=\zeta=0} \,. \tag{2.4}$$

The non-commutative analog of the classical massive scalar field theory action (without derivative couplings) on a commutative space reads

$$\tilde{S}[\Phi] = \mathrm{tr}\left(\sum_\mu \left(\frac{1}{2}\theta^{-1}_{\mu\nu}[\hat{x}^\nu, \Phi(\hat{x})]\right)^2 + \frac{m^2}{2}\Phi(\hat{x})^2 + \frac{g}{n}\left(\Phi(\hat{x})\right)^n\right). \tag{2.5}$$

This action can be expressed in terms of $\phi(x)$ defined by eq. (2.3) as

$$S[\phi] = \int d^d x \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{n}\left(\phi \star \cdots \star \phi\right)\right]. \tag{2.6}$$

The action eq. (2.6) in the momentum space reads

$$S[\phi] = \int d^d k \, \frac{1}{2}\phi(-k)\left(k^2 + m^2\right)\phi(k) + \\ + \int d^d k_1 \cdots d^d k_n V(k_1, \ldots, k_n)\, \phi(k_1) \cdots \phi(k_n)\,, \tag{2.7}$$

where

$$V(k_1, \ldots, k_n) = \frac{1}{n}\delta(k_1 + \cdots + k_n) \exp\left(\sum_{i<j} k_i^\mu k_j^\nu \theta_{\mu\nu}\right). \tag{2.8}$$

Due to the noncommutativity of the $\star$-product, the interaction term in $S[\phi]$ is not totally symmetric under the exchange of the arguments, but only under cyclic permutations. This implies that the Feynman graphs of NQFT are equivalent to ribbon graphs. Thus, a general diagram of NQFT can be drawn on a genus g 2-surface. A NQFT Feynman graph may have crossings of internal and external lines. In order to find the contribution of the phase factors eq. (2.8) to an arbitrary Feynman graph G, one may find it useful first to apply the following operations to G [5] (the same set of operations were first introduced in a different context in ref. [17]):

(1) Contraction of two vertices connected by a line (figure 3):

$$V\left(k_1, \ldots, k_{n_1}, p\right) V\left(-p, k_{n_1+1}, \ldots, k_{n_2}\right) = V\left(k_1, \ldots, k_{n_2}\right) \delta\left(k_1 + \cdots + k_{n_1} + p\right).$$

(2) Elimination of a loop which does not cross other lines (figure 4):

$$V\Big(k_1, \ldots, k_{n_1}, p, k_{n_1+1}, \ldots, k_{n_2}, -p\Big) = \\ = V\Big(k_1, \ldots, k_{n_1}, k_{n_1+1}, \ldots, k_{n_2}\Big) \quad \text{if} \sum_{i=n_1+1}^{n_2} k_i = 0\,.$$

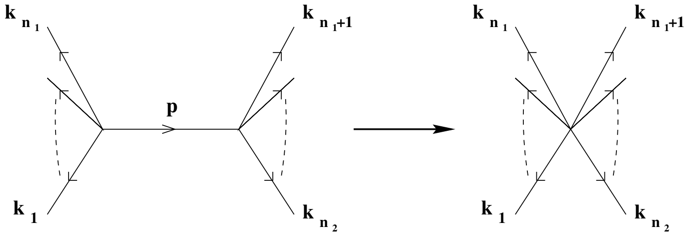


Figure 3: Filk's operation 1.

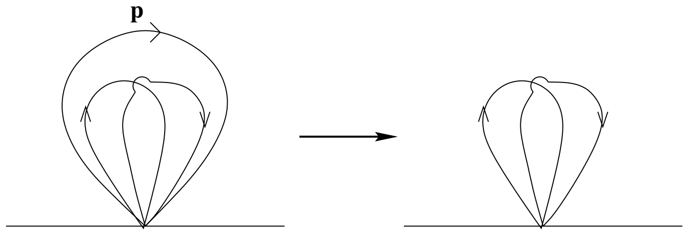


Figure 4: Filk's operation 2.

These operations are based on momentum conservation and cyclic symmetry at each vertex. Using these two operations one may reduce any Feynman graph to a graph which consists of only one vertex.

For a planar graph this reduction leads to a one vertex graph with external lines. For a non-planar graph this reduction leads to a rosette.[10] The set of rosette lines of a graph G is denoted as $\mathcal{R}(G)$.

Example 2 Shrinking lines 2, 7, 6, 5 and then 3, 9 in figure 2 a, one finds the rosette of figure 5. In this case $\mathcal{R} = \{4, 8\}$.

Let us define $I_{ij}(G)$ to be the intersection matrix of internal lines of an oriented graph G (orientation is given by the sign convention chosen for the momenta in the conservation conditions): if $i, j \in \mathcal{R}(G)$, then

$$I_{ij}(G) = \begin{cases} 1 & \text{line } j \text{ crosses } i \text{ from right}\,, \\ -1 & \text{line } j \text{ crosses } i \text{ from left}\,, \\ 0 & \text{line } j \text{ and } i \text{ do not cross}\,. \end{cases} \tag{2.9}$$

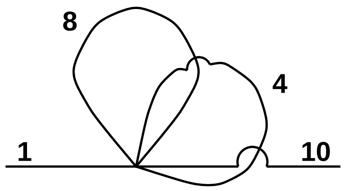


Figure 5: Illustration for example 2.

Otherwise, $I_{ij} = 0$. Note that $I_{ij} = -I_{ji}$. Similarly, one can define the intersection matrix J_{mv} of internal and external lines.

We shall now consider an arbitrary non-commutative graph G in scalar NQFT eq. (2.6) and compute the corresponding contribution I_G as given by non-commutative Feynman rules. We assume that G has no tadpoles. $\mathcal{L}(G)$ and $\mathcal{V}(G)$ denote the set of lines and vertices of the graph G respectively. I and V denote the number of lines and vertices of G, respectively. Define the incidence matrix $\{\epsilon_{vl}\}$, with indices

[10]Different reductions may give different rosettes, but all of them give the same phase factor.

running over vertices and internal lines, respectively, as

$$\epsilon_{vl} = \begin{cases} 1 & \text{if the vertex } v \text{ is the starting point of the line } l\,, \\ -1 & \text{if the vertex } v \text{ is the endpoint of the line } l\,, \\ 0 & \text{if } l \text{ is not incident on } v\,. \end{cases} \tag{2.10}$$

Let us denote by P_v the total external momentum flowing into the vertex v. With these conventions I_G reads

$$\begin{aligned} I_G(P) = \int \prod_{l=1}^{I} d^d k_l \left(\frac{1}{k_l^2+m_l^2}\right) \prod_{v=1}^{V} \left[(2\pi)^d \delta^{(d)}\left(P_v - \sum_l \epsilon_{vl}\, k_l\right)\right] \times \\ \times \exp\left[i\left(\sum_{m,n} I_{mn}\,\theta_{\mu\nu}\,k_m^\mu\,k_n^\nu + \sum_{m,v} J_{mv}\,\theta_{\mu\nu}\,k_m^\mu\,P_v^\nu\right)\right]. \end{aligned} \tag{2.11}$$

2.2 Parametric integral representation and topological formula

The parametric integral representation of eq. (2.11) is derived in appendix A and it reads

$$\begin{aligned} I_G(P) = 2^d \left(\sqrt{\pi}\right)^{(I+V+1)d} \delta^{(d)}\left(\sum_v P_v\right) \int_0^\infty \prod_{l=1}^{I} d\alpha_l \frac{e^{-\sum_l \alpha_l m_l^2}}{\sqrt{\det\mathcal{A}\,\det\mathcal{B}}} \times \\ \times \exp\left\{\frac{1}{4}\left[\epsilon\mathcal{A}^{-1}(J\eta)+2iP\right]_v^\mu (\mathcal{B}^{-1})_{v\tilde{v}}^{\mu\nu} \left[\epsilon\mathcal{A}^{-1}(J\eta)+2iP\right]_{\tilde{v}}^\nu - \right. \\ \left. -\frac{1}{4}(J\eta)_m^\mu (\mathcal{A}^{-1})_{mn}^{\mu\nu}(J\eta)_n^\nu\right\}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \mathcal{A}_{mn}^{\mu\nu} \equiv \alpha_m\delta_{mn}\delta_{\mu\nu} - iI_{mn}\theta_{\mu\nu}\,, \qquad \mathcal{B}_{v\tilde{v}}^{\mu\nu} = \epsilon_{vm}(\mathcal{A}^{-1})_{mn}^{\mu\nu}\epsilon_{\tilde{v}n}\,, \quad v,\tilde{v} = 1,\ldots,V-1\,, \\ (J\eta)_m^\mu \equiv \sum_{v=1}^{V} J_{mv}\eta_v^\mu\,, \quad \eta_v^\mu \equiv \theta_{\mu\nu}P_v^\nu\,, \qquad (y\epsilon)_m^\mu \equiv \sum_{v=1}^{V} y_v^\mu \epsilon_{vm}\,. \end{aligned} \tag{2.13}$$

Without loss of generality one may assume that the matrix θ is in the Jordan form

$$\theta = \begin{pmatrix} 0 & -\theta_1 & & & \\ \theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_{d/2} \\ & & & \theta_{d/2} & 0 \end{pmatrix}. \tag{2.14}$$

Let r be the rank of the intersection matrix I_{ij}. The structure of the pre-exponential factor in eq. (2.12) is then

$$\sqrt{\det\mathcal{A}\,\det\mathcal{B}} = \prod_{i=1}^{d/2}\left(\sum_{n=0}^{\tilde{g}} \theta_i^{2n} P_{2n}(\alpha)\right), \tag{2.15}$$

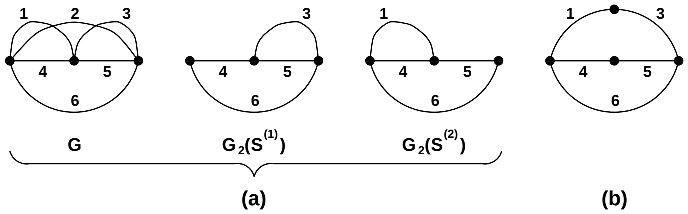


Figure 6: Illustration for examples 3 and 5.

where $\tilde{g} = r/2$ is defined in terms of the cycle number $\tilde{c}(G)$[11] as

$$\tilde{g} = \frac{\tilde{c}(G)}{2} \tag{2.16}$$

and $P_{2n}(\alpha)$ is a sum of monomials of degree $L - 2n$:

$$P_{2n} = \sum_{\{i_1, i_2, \ldots, i_{L-2n}\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{L-2n}} \,. \tag{2.17}$$

Note that the coefficient in front of each monomial is one.

We now give a topological formula for P_{2n}. Let $\mathcal{S}_{2n} = \{i_1, \ldots, i_{2n}\} \subset \mathcal{R}(G)$ be a set consisting of $2n$ linearly independent lines from $\mathcal{R}(G)$, i.e. the intersection matrix I_{ij} restricted to the lines $i_1, \ldots, i_{2n}$ is non degenerate. For this set $\mathcal{S}_{2n}$, one can define the graph $G_{2n}(\mathcal{S})$ obtained from the graph G by deleting the lines of $\mathcal{S}_{2n}$. Thus $\mathcal{L}(G_{2n}(\mathcal{S})) = \mathcal{L}(G) \setminus \mathcal{S}_{2n}$ and $\mathcal{V}(G_{2n}(\mathcal{S})) = \mathcal{V}(G)$. For a given graph G and rosette $\mathcal{R}(G)$, there can be several different $\mathcal{S}_{2n}$: $\mathcal{S}_{2n}^{(1)}, \ldots, \mathcal{S}_{2n}^{(m)}$. For each $\mathcal{S}_{2n}^{(k)}$ one has a graph $G_{2n}(\mathcal{S}^{(k)})$. For a given graph γ define the so-called chord-set product sum

$$C(\alpha, \gamma) = \sum_{\ell \in \mathcal{T}^*(\gamma)} \prod_{l \in \ell} \alpha_l \,, \tag{2.18}$$

where $\mathcal{T}^*(\gamma)$ is the set of all chords of the graph γ [18].

Example 3 For the graph G in figure $6a$, $\mathcal{S}_2^{(1)} = \{1, 2\}$, $\mathcal{S}_2^{(2)} = \{2, 3\}$, and

$$C\left(\alpha, G_2\left(\mathcal{S}^{(1)}\right)\right) = (\alpha_3 + \alpha_5)(\alpha_4 + \alpha_6) + \alpha_3\alpha_5 \,,$$
$$C\left(\alpha, G_2\left(\mathcal{S}^{(2)}\right)\right) = (\alpha_1 + \alpha_4)(\alpha_5 + \alpha_6) + \alpha_1\alpha_4 \,.$$

Let us define the addition $\oplus$ on the space of homogeneous polynomials of a given degree in α's and with unit coefficients. Let $P_{2n}(\alpha)$ and $Q_{2n}(\alpha)$ be two such polynomials. Then we define

$$P_{2n} \oplus Q_{2n} \equiv P_{2n} + Q_{2n} \pmod 2 \,.$$

[11] For the definition of $\tilde{c}(G)$ see section 3. $\tilde{g}$ is simply the genus of the graph G at vanishing external momenta.

Example 4

$$(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_4\alpha_5) \oplus (\alpha_1\alpha_2 + \alpha_4\alpha_5) \oplus (\alpha_4\alpha_5 + \alpha_1\alpha_6) = \alpha_2\alpha_3 + \alpha_4\alpha_5 + \alpha_1\alpha_6 \,.$$

With these conventions and definitions, the following theorem holds

Theorem 1 (Topological formula)

$$P_{2n}(\alpha, G) = \bigoplus_{k=1}^{m} C\left(\alpha, G_{2n}\left(\mathcal{S}^{(k)}\right)\right). \tag{2.19}$$

The proof of this theorem is somewhat technical and will not be given here. Note that for $n = 0$, eq. (2.19) reads as $P_0(\alpha, G) = C(\alpha, G)$.

Example 5 For the graph G in figure 6*a*, we have

$$P_2 = (\alpha_1 + \alpha_3)(\alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)\,. \tag{2.20}$$

Note that P_2 in eq. (2.20) is equal to the chord-set product sum eq. (2.18) for the graph in figure 6*b*.

3. Convergence theorem

This section is organized as follows. In subsection 3.1 we prove the convergence of Feynman integrals for some classes of graphs in the massive scalar NQFT. In subsection 3.2 we formulate a general convergence theorem for non-commutative Feynman graphs and demonstrate it on the graphs discussed in subsection 3.1.

3.1 Examples and propositions

In eq. (2.12) the UV divergences show up as poles at $\alpha = 0$ of the integrand. The integral eq. (2.12) is convergent at the upper limit of the integration because there are no IR divergences in the massive theory. Let us consider the diagram in figure 7*a*. In the commutative limit it is quadratically divergent in six dimensions. But as a non-commutative graph, it has a crossing of internal lines and it is a genus $g = 1$ graph. In d dimensions, the prefactor of the exponent in eq. (2.12) for this graph reads

$$\sqrt{\det\mathcal{A}\det\mathcal{B}} = \prod_{i=1}^{d/2}\left[(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \theta_i^2\right]. \tag{3.1}$$

Due to the non-zero θ's, eq. (3.1) has no zeros in the range of integration of eq. (2.12). Thus the graph in figure 7*a* is convergent. It is easy to see from the α-representation eq. (2.12) that at large-external momentum k it behaves as $\sim 1/k^{10} = k^{\omega - 2gd}$ in any dimension d. Note that the graph in figure 5*a* is an example of the graphs for which the number of lines in $\mathcal{R}(G)$ *equals* $2g$. The following proposition is true for such graphs.

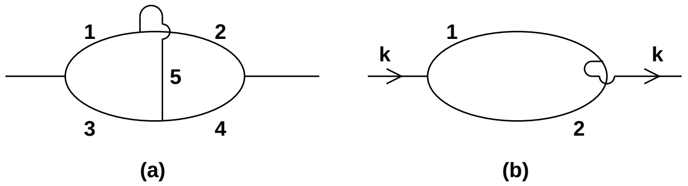


Figure 7: Two basic examples of graphs regulated by the noncommutativity.

Proposition 1 *If the intersection matrix $I_{ij}(G)$ restricted to the lines of the rosette $\mathcal{R}(G)$ is non degenerate and $I_{G-\mathcal{R}(G)}$ for the planar graph $G-\mathcal{R}(G)$ converges, then I_G converges.*

Proof. Let $r = 2g$ be the rank of the matrix I_{ij}. Since I_{ij} is non degenerate when restricted to the rosette, $m = 1$ in eq. (2.19). This implies that $P_{2g}(G) = P_0(G - \mathcal{R}(G))$. Since

$$\frac{1}{\prod_{i=1}^{d/2}\left(\sum_{n=0}^{g}\theta_i^{2n}P_{2n}(\alpha)\right)} \leq \frac{1}{\prod_{i=1}^{d/2}\left(\theta_i^{2g}P_{2g}(\alpha)\right)}$$

we conclude that I_G converges if $I_{G-\mathcal{R}(G)}$ converges. □

Now consider the graph in figure 7*b*. It has an intersection of the internal line with the external line. eq. (2.12) for this graph yields

$$\int_0^\infty d\alpha_1\, d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{d/2}} \exp\left(-(\alpha_1+\alpha_2)m^2 - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}k^2 - \frac{|\theta k|^2}{4(\alpha_1+\alpha_2)}\right). \quad (3.2)$$

Note that in eq. (3.2), there is a term proportional to $1/\alpha$ in the exponent. This is a general property of graphs with external lines crossing internal lines. The $1/\alpha$ terms in the exponent will suppress divergences coming from the pre-exponential factor. The following proposition is true for this type of graphs.

Proposition 2 *If the non-planarity of a graph G is solely due to the intersection of an external line with one internal line as in figure 8, i.e. $J_{mv} = \delta_{mm_0}\,\delta_{vv_0}$ and $I_{ij} \equiv 0$, and for any 1PI subgraph $\gamma \subset G$* not containing line m_0 *one has $\omega(\gamma) < 0$, then I_G converges.*

Proof. At $\alpha_1 = \cdots = \alpha_I = t \sim 0$, various terms in the exponent of eq. (2.12) scale as follows. The J^0 term scales like $O(t)$. The J^1 term gives rise to an oscillatory contribution and thus cannot suppress the divergence at $\alpha \sim 0$. The J^2 term scales like $O(1/t)$. We thus consider only J^2 term.

Since $I_{ij} = 0$ we have $(\mathcal{A}^{-1})^{\mu\nu}_{mn} = \frac{1}{\alpha_n}\delta_{mn}\,\delta_{\mu\nu}$. Let us choose V in eq. (2.13) as in figure 8. Simple algebra gives

$$(J\eta)^\mu_m\,(\mathcal{A}^{-1})^{\mu\nu}_{mn}\,(J\eta)^\nu_n = \frac{|\eta_{v_0}|^2}{\alpha_{m_0}},$$

$$\left[\epsilon\mathcal{A}^{-1}(J\eta)\right]^{\mu}_{v}(\mathcal{B}^{-1})^{\mu\nu}_{v\tilde{v}}\left[\epsilon\mathcal{A}^{-1}(J\eta)\right]^{\nu}_{\tilde{v}} = \frac{|\eta_{v_0}|^2}{\alpha^2_{m_0}}\mathcal{B}^{-1}_{v_0 v_0}\,. \tag{3.3}$$

Using

$$\mathcal{B}^{-1} = \frac{1}{\det\mathcal{B}}\det\mathcal{B}|_{v_0}\ \text{deleted} \tag{3.4}$$

and the fact that deleting v_0 is equivalent to shrinking the line m_0, one finds

$$\mathcal{B}^{-1}_{v_0 v_0} = \frac{\alpha_{m_0}P_0(G/m_0)}{P_0(G)}\,, \tag{3.5}$$

where G/m_0 denotes the graph obtained from G by shrinking the line m_0.

Using eqs. (3.3) and (3.5) and the relation

$$P_0(G) = \alpha_{m_0}P_0(G-m_0) + P_0\left(\frac{G}{m_0}\right) \tag{3.6}$$

it is easy to show that the J^2 term in eq. (2.12) gives the following contribution:

$$\exp\left(-\frac{|\eta_{v_0}|^2}{4}\frac{P_0(G-m_0)}{P_0(G)}\right).$$

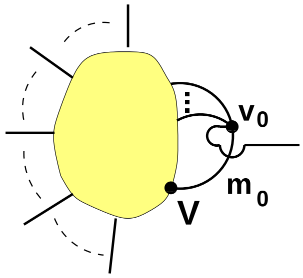


Figure 8: Illustration for the proposition 2.

Following ref. [18], let us divide the integration domain in eq. (2.12) into sectors

$$0 \le \alpha_{\pi_1} \le \alpha_{\pi_2} \le \cdots \le \alpha_{\pi_I}\,,$$

where π is a permutation of $(1,2,\ldots,I)$. To each sector corresponds a family of nested subsets γ_l of lines of G:

$$\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_I = G\,,$$

where γ_l contains the lines pertaining to $(\alpha_{\pi_1},\ldots,\alpha_{\pi_l})$. In the sector given by π, perform a change of variables

$$\begin{aligned}
\alpha_{\pi_1} &= \beta_1^2\beta_2^2\cdots\beta_s^2\cdots\beta_{I-1}^2\beta_I^2\,,\\
\alpha_{\pi_2} &= \quad\beta_2^2\cdots\beta_s^2\cdots\beta_{I-1}^2\beta_I^2\,,\\
&\ \vdots\\
\alpha_{\pi_s} &= \qquad\quad\beta_s^2\cdots\beta_{I-1}^2\beta_I^2\,,\\
&\ \vdots\\
\alpha_{\pi_{I-1}} &= \qquad\qquad\quad\beta_{I-1}^2\beta_I^2\,,\\
\alpha_{\pi_I} &= \qquad\qquad\qquad\quad\beta_I^2
\end{aligned} \tag{3.7}$$

the jacobian of which is

$$\frac{D(\alpha_1,\ldots,a_I)}{D(\beta_1,\ldots,\beta_I)} = 2^I\beta_1\beta_2^3\cdots\beta_I^{2I-1}\,.$$

In these β variables the integration domain in eq. (2.12) reads

$$0 \leq \beta_I \leq \infty \qquad \text{and} \qquad 0 \leq \beta_l \leq 1 \quad \text{for } 1 \leq l \leq I-1\,.$$

Let L_l be the number of independent loops in γ_l. It can be shown that [18]

$$P_0(G) = \beta_1^{2L_1}\beta_2^{2L_2}\cdots\beta_I^{2L_I}[1+O(\beta)]\,. \tag{3.8}$$

We derive a similar relation for $P_0(G-m_0)$ as follows. Suppose that $\pi_s = m_0$. Then the graphs $\gamma_1,\ldots,\gamma_{s-1}$ do not contain line m_0. The graph γ_s contains line m_0, but it may happen that one end of m_0 is free, i.e. m_0 is attached to γ_{s-1} with only one end. Let $\gamma_k,\ k \geq s$ be the first graph for which both ends of m_0 are not free. By inspection of eq. (3.7) it is not difficult to see that

$$P_0(G-m_0) = \beta_1^{2L_1}\beta_2^{2L_2}\cdots\beta_{k-1}^{2L_{k-1}}\beta_k^{2(L_k-1)}\cdots\beta_I^{2(L_I-1)}\,. \tag{3.9}$$

The leading term at $\alpha \sim 0$ in the integrand of eq. (2.12) is then

$$\begin{aligned}&\frac{\beta_1^1\beta_2^3\cdots\beta_I^{2I-1}}{\beta_1^{dL_1}\beta_2^{dL_2}\cdots\beta_I^{dL_I}}\exp\left(-\frac{|\eta_{v_0}|^2}{4}\frac{1}{\beta_k^2\cdots\beta_I^2}\right) = \\ &= \left(\prod_{l=1}^{k-1}\beta_l^{-\omega_l-1}\right)\left(\prod_{l=k}^{I}\beta_l^{-\omega_l-1}\right)\exp\left(-\frac{|\eta_{v_0}|^2}{4}\frac{1}{\beta_k^2\cdots\beta_I^2}\right),\end{aligned} \tag{3.10}$$

where $\omega_l = dL_l - 2l$. Since $\omega_l < 0$ for $l < k$, the integral eq. (2.12) converges. **q.e.d.**

Let us mention a peculiar feature of the diagrams with external lines crossing internal lines (figure 7*b*). At large external momenta they scale like $\exp(-\text{const.}k^2\theta)$.[12] But once such a graph is put inside a bigger graph, it behaves as if it has dimension $\omega - d$.

Example 6 The subgraph γ formed by lines 1 and 4 of graph G in figure 6*a* has line 2 as an external line crossing the internal line 1. A simple rescaling $\alpha_1, \alpha_4 \to \rho\alpha_1, \rho\alpha_4$ in

$$\frac{d\alpha_1\cdots d\alpha_6}{(P_0+\theta^2P_2)^{d/2}}\,,$$

where P_2 is given by eq. (2.20), shows that the subgraph γ behaves as $k^{\omega(\gamma)-d}$ when its external momenta are large.

This circumstance of a graph behaving differently in different contexts makes it difficult to implement the approach of the asymptotic algebra developed in refs. [20, 21] for the analysis of the asymptotic behavior of usual Feynman diagrams to our case. Presumably, one may find an algebra of asymptotic functions in the NQFT case and use it for an inductive proof of the convergence theorem.

In the remaining example and proposition of this subsection we will need the following lemma.

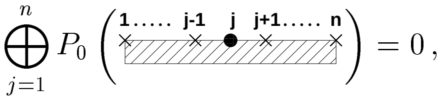
[12] This UV behavior of diagrams may have relation to the UV behavior of gauge non-invariant correlators of large-N non-commutative SYM theory calculated in ref. [14] using supergravity/gauge theory correspondence.

JHEP05(2000)037

Lemma 1 *Let us choose n vertices of a graph G and identify $n-1$ of them. Let j be the remaining vertex. Denote by G_j the resulting graph. Letting j to run from 1 to n one finds different G_j's. Then the following relation holds*

$$\bigoplus_{j=1}^{n} P_0\left(G_j\right) = 0\,.$$

Pictorially, it reads

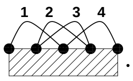


(3.11)

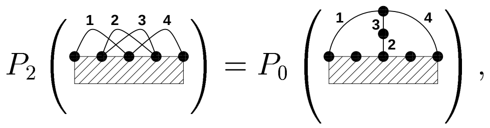
where the crosses $\times$ denote identified vertices.

The proof of this lemma is given in appendix B.

The relations proved in the following example and proposition will be used in subsection 3.2 for the analysis of the convergence properties of the graphs.

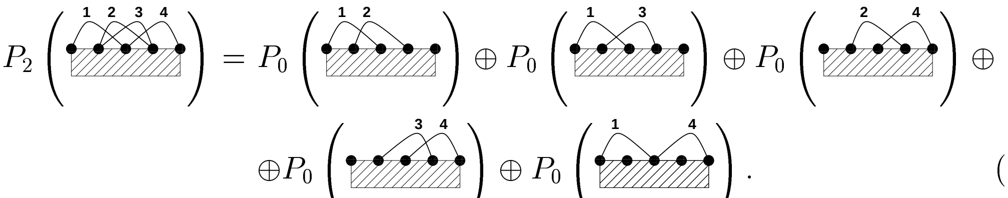
Example 7 Consider the graph G_1: 

. Let us prove that



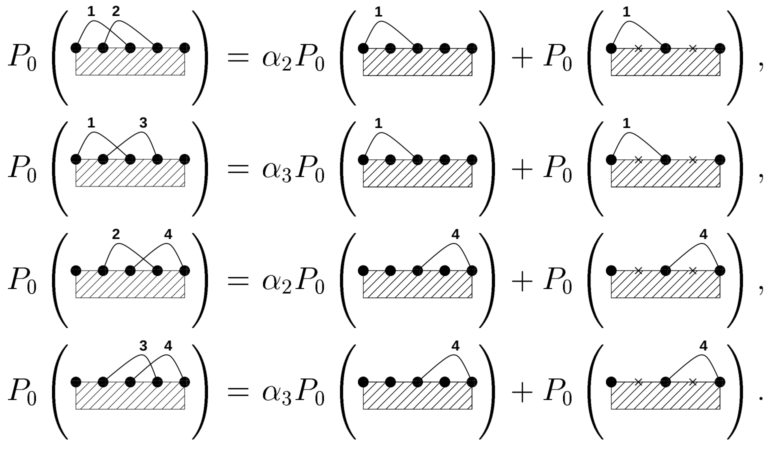
(3.12)

where the hashed block denotes an arbitrary planar subgraph. G_1 is a $g=1$ graph and it is not of the type considered in proposition 1. Applying theorem 1 to the l.h.s. of eq. (3.12) one finds



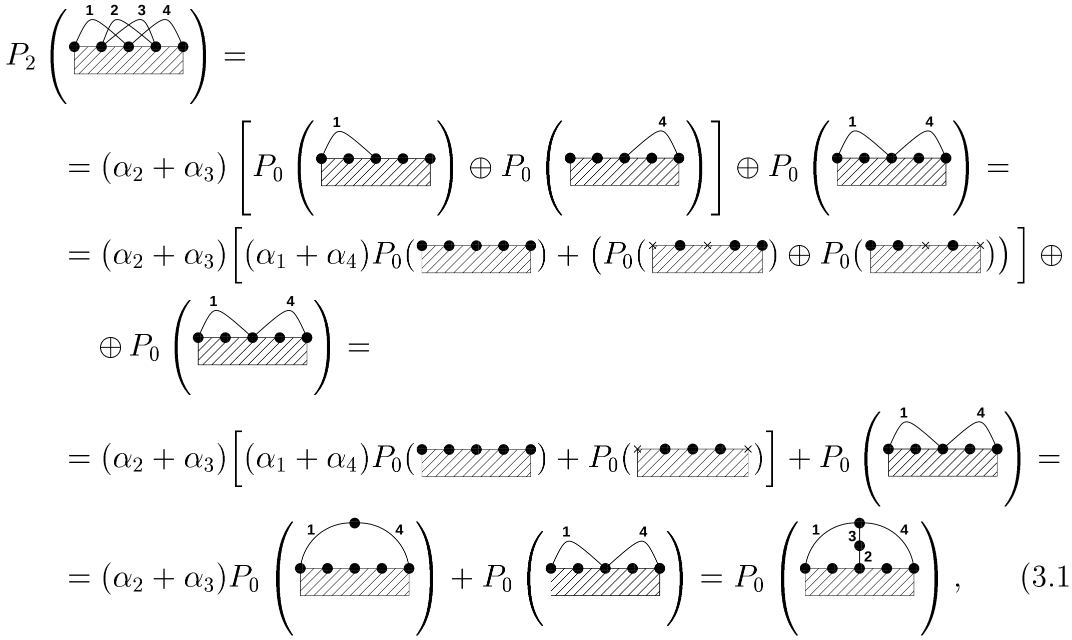
(3.13)

Let G/l and $G-l$ be the graphs obtained from G by shrinking and deleting the line l, respectively. Using the general relation $P_0(G) = \alpha_l P_0(G-l) + P_0(G/l)$, we have



(3.14)

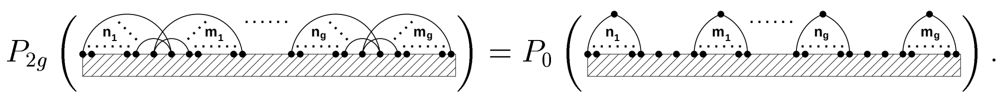
Using eq. (3.14) and the definition of $\oplus$ one finds



(3.15)

where for the third equality we used the lemma. Thus we have proven eq. (3.12). Equation (3.12) will be used in section 3.2 for the analysis of the convergence property of G_1. The reader may try to prove similar relations for various graphs involving several crossing lines. One can even prove some quite general relations as in the following proposition.

Proposition 3



(3.16)

This proposition can be proven by induction using the Lemma along the lines of the proof given in the example 7.

A remarkable feature of these relations is that they relate P_{2g} of a genus g graph to P_0 of a genus zero graph. This suggests that there should exist a *natural* map

$$\mathcal{F}_{g_1\to g_2} : \mathcal{G}_{g_1} \longrightarrow \mathcal{G}_{g_2}\,, \quad g_1 > g_2$$

between sets $\mathcal{G}_{g_1}$, $\mathcal{G}_{g_2}$ of graphs of genera g_1, g_2, such that the relation

$$P_{2g_1}\left(G_{g_1}\right) = P_{2g_2}\left(\mathcal{F}_{g_1\to g_2}\left(G_{g_1}\right)\right),$$

where $G_{g_1} \in \mathcal{G}_{g_1}$, holds.

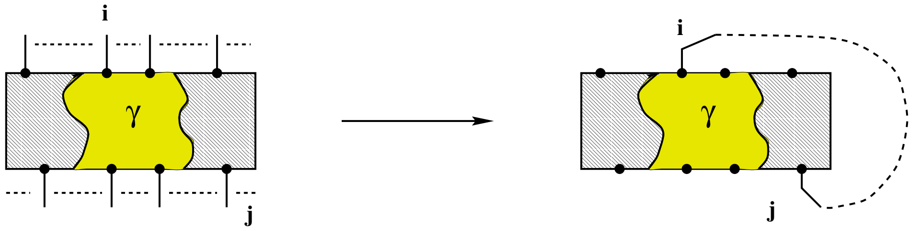


Figure 9: Definition of index j.

3.2 Convergence theorem and analysis of various graphs

In this subsection we formulate a general convergence theorem for the non-commutative graphs and illustrate it on the graphs considered in section 3.1. Let us give several definitions required for the formulation of the theorem. Let G be a genus g graph with a set of external lines $\mathcal{E}(G)$. Such a graph can be drawn on a genus g 2-surface, Σ_g, with a boundary, $\partial\Sigma_g$, to which the external lines are attached.

Let us define a cycle number $c(\gamma)$ for the subgraph $\gamma \subset G$. The first homology group of Σ_g for the graph G has the basis $\mathcal{C} = \{a_1, b_1, \ldots, a_g, b_g\}$ (see figure 1). One may go to a different basis by forming combinations of the elements of $\mathcal{C}$. $c(\gamma)$ is defined as the number of inequivalent non-trivial cycles of Σ_g spanned by the closed paths in γ. The following example illustrates the definition of c.

Example 8 Let us denote the hashed planar part of graph G_1 in example 7 by γ. If we draw G_1 on a $g = 1$ surface, we see that $c(\gamma \cup \{i\}) = 1, i = 1, \ldots, 4$.

Let us consider now a graph G at the vanishing $p = 0$ external momenta. It can be drawn on a genus g' 2-surface $\Sigma_{g'}^{p=0}$. In general $\Sigma_{g'}^{p=0}$ is different from Σ_g. Let us define the cycle number $\tilde{c}(\gamma)$ of the subgraph $\gamma \subset G$ to be the number of inequivalent non-trivial cycles of $\Sigma_{g'}^{p=0}$ spanned by the closed paths in γ. In general $c(\gamma) \neq \tilde{c}(\gamma)$.

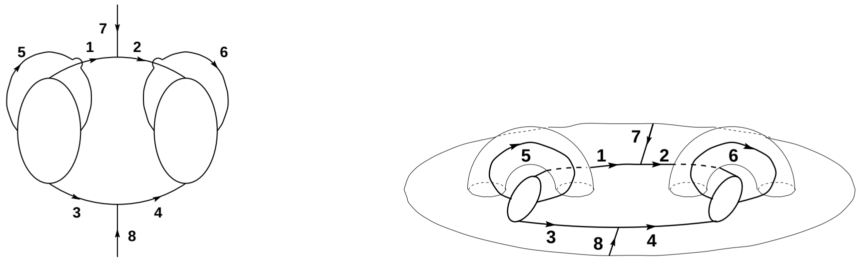
Consider two external lines $i, j \in \mathcal{E}(G)$. As in figure 9, set the rest of the external momenta graph G to zero and connect the lines i and j. Let $c_{ij}(\gamma)$ be the cycle number of an arbitrary subgraph $\gamma \subset G$ with respect to the 2-surface of the resulting graph. There are only two possibilities:

$$c_{ij}(\gamma) > \tilde{c}(\gamma) \qquad \text{or} \qquad c_{ij}(\gamma) = \tilde{c}(\gamma)\,.$$

We define the index j of an arbitrary (connected or disconnected) subgraph $\gamma \subset G$ as follows: if there *exists* a pair of external lines i, j such that $c_{ij}(\gamma) > \tilde{c}(\gamma)$, then $j(\gamma) = 1$. Otherwise, $j(\gamma) = 0$. The following examples illustrate this definition.

Example 9a Consider the following diagram:

This graph has genus $g = 1$ and $c(G) = 1$. By setting the external momenta to zero we get $g' = 0$ and $\tilde{c}(G) = 0$. By joining line 2 with line 1 we get $c_{21}(G) = 1$ (we also have $c_{23}(G) = 1$). Thus, according to our definition, $j(G) = 1$. Let us analyze the subgraphs of G:



(a) (b)

Figure 11: Illustration for the example 9a.

- $\gamma_1 = \{5, 6, 7\}$: this subgraph has $c_{21}(\gamma_1) = c_{23}(\gamma_1) = 1$ and $\tilde{c}(\gamma_1) = 0$ which implies $j(\gamma_1) = 1$.

- $\gamma_2 = \{4, 7, 8, 9\}$: this subgraph does not wrap any non-trivial cycle; $c(\gamma_2) = 0 = \tilde{c}(\gamma_2)$ and thus $j(\gamma_2) = 0$.

- $\gamma_3 = \{4, 5, 6, 8, 9\}$: this subgraph has properties similar to γ_1: $c_{21}(\gamma_3) = c_{23}(\gamma_3) = 1$ and $\tilde{c}(\gamma_3) = 0$ and therefore $j(\gamma_3) = 1$.

Example 9b Consider the graph in figure 10. It has genus $g = 2$ and $c(G) = 3$. This is also the g and c for the graph obtained by joining the external lines. It is easy to see that by setting the external momenta to zero the genus becomes $g' = 1$ and $\tilde{c}(G) = 2$. We therefore conclude that $j(G) = 1$.

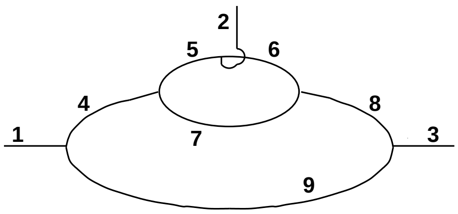
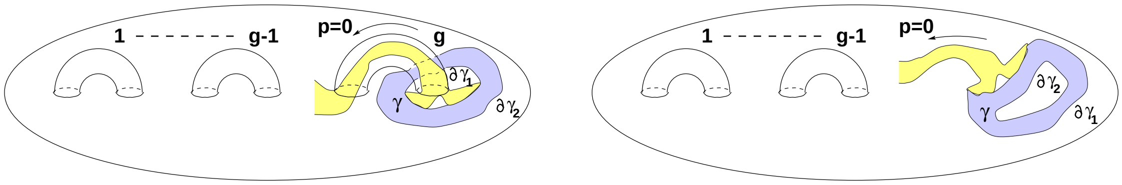


Figure 10: Illustration for the example 9b.

Let us explain the difference between homologically trivial and non-trivial cycles from the point of view of the momentum flow on the surface Σ_g of a graph. Figure 1b illustrates the flow of momentum on a genus two surface with a boundary. There are *topologically trivial* flows like p_0 and *topologically non-trivial* flows like p_A, p_C, p_{a_2} and p_{b_2}. Since the total external momentum flowing into the surface Σ_2 through $\partial\Sigma_2$ is zero, the net momentum flowing across A and C is zero (the momenta p_A, p_C along A and C are in general non zero). The phase factor associated with a graph arises from the *linking* of topologically non-trivial flows. In figure 1b, p_{a_2} and p_{b_2} contribute a phase factor $\exp(i\theta_{\mu\nu}p^{\mu}_{a_2}p^{\nu}_{b_2})$. Cycles A and C do not contribute to the phase factor because the net momentum flowing across each of these cycles is zero. Since the cycles a_2, b_2 are homologically nontrivial and the cycles A, C and 0 are homologically trivial, we conclude that *only the momentum flow along the homologically non-trivial cycles contribute to the phase factor.*



(a) (b)

Figure 12: Two equivalent graphs.

Example 10a The total momentum flowing along the cycle a of the graph in figure 2b is $p_a = q_8 + q_{10}$. The total momentum flowing along the cycle b is $p_b = q_8 + q_{10} - q_4$. Thus the phase factor is

$$\exp\left(i\theta_{\mu\nu}p_a^\mu p_b^\nu\right) = \exp\Big(i\theta_{\mu\nu}q_4^\mu(q_8+q_{10})^\nu\Big).$$

Example 10b In figure 11b $p_{b_1} = q_5$, $p_{b_2} = q_6$, $p_{a_1} = -q_1$ and $p_{a_2} = q_2$. Thus the phase factor is

$$\exp\Big(i\theta_{\mu\nu}\left(q_5^\mu q_1^\nu + q_2^\mu q_6^\nu\right)\Big).$$

One might object that the statement made above regarding the homologically non-trivial cycles is not always true by giving the following counter-example. In figure 12a a graph G is drawn on a genus g surface with a boundary. The subgraph γ wraps the cycle a_g and it is connected to the rest of the diagram only through the handle g. Due to the momentum conservation, the net momentum flowing along the cycle b_g is zero. Thus there is no phase factor associated with γ. The subgraph γ seems to be homologically nontrivial, but there is no phase factor associated with it. The point is that one can "slide" γ through the handle g and redraw the 2-surface as in figure 12b.

The resulting surface has genus $g-1$. Considered as non-commutative Feynman graphs, the graphs in figure 12a and figure 12b are the same graph, i.e. it appears only once in the perturbative expansion.

Theorem 2 (Convergence theorem) *In a massive NQFT in d dimensions, a graph 1PI G is convergent if and only if for any subgraph*[13] $\gamma \subseteq G$ *at least one of the following conditions is satisfied:*

1. $\omega(\gamma) - c(\gamma)d < 0$.

2. $j(\gamma) = 1$.

An inductive proof of this theorem will be given in ref. [19].

[13] In particular γ can be a disjoint union of 1PI subgraphs of G. It is assumed that the external momenta of the graph G are generic. The integral may diverge for exceptional external momenta which form a set of measure zero in the space of external momenta.

JHEP05(2000)037

It is not difficult to see from eqs. (2.12), (2.15), (2.17) and the relation

$$P_{2n}(G) = P_{2n}(\gamma) P_0 \left(\frac{G}{\gamma}\right) + X$$

(X is a sum of terms whose degree with respect to α_l ($l \in \gamma$) is at least $L(\gamma) - 2n + 1$), that the conditions (1) and (2) are necessary. The non-trivial part of the convergence theorem is the *sufficiency* of the conditions (1)–(2).

We now demonstrate that theorem 2 holds for the graphs we considered in previous sections.

Analysis of figure 6. The condition (1) is satisfied for any subgraph of G if $d < 6$. Using the relation $1/(P_0 + \theta^2 P_2) \leq 1/\theta^2 P_2$ and the fact that $P_2(G)$ equals P_0 of the graph shown in figure 6*b*, we see that I_G indeed converges if $d < 6$.

Analysis of proposition 1. Proposition 1 states that I_G converges if $I_{G-\mathcal{R}(G)}$ converges. Let us see how this follows from theorem 2. The subgraph $G - \mathcal{R}(G)$ is planar and according to the condition (1) of theorem 2 it should satisfy $\omega(G - \mathcal{R}(G)) < 0$. Since

$$\omega(G) = \omega\left(G - \mathcal{R}(G)\right) + (d-2) I(\mathcal{R}) = \omega\left(G - \mathcal{R}(G)\right) + 2\,g(d-2)$$

the condition $\omega(G) - 2gd < 0$ is satisfied. One can similarly show that for any non-planar subgraph of G the condition (1) is satisfied if $\omega < 0$ for all planar subgraphs of G.

Analysis of proposition 2. Choose the decomposition of the external lines $\mathcal{E}(G) = \{P_{v_0}\} \cup \{\text{rest}\}$. Thus $j(G) = 1$. Any subgraph of G not containing the external line P_{v_0} has $j = 0$ and thus should satisfy $\omega < 0$. Any subgraph of G which contains P_{v_0} and at least one other external line of G has $j = 1$. Such a subgraph may have $\omega \geq 0$, but it satisfies the condition (2) of the convergence theorem.

Analysis of example 7. Let us denote the hashed block of G_1 as γ. From the relation $\omega(G_1) = \omega(\gamma) + 4(d-2)$, we see that the condition (1) for G_1: $\omega(G_1) - 2d < 0$ is satisfied if $\omega(\gamma) < 8 - 2d$. Let us see how the same conclusion follows from eq. (3.12). Equation (3.12) states that $P_2(G_1) = P_0(G_0)$. Thus I_{G_1} converges if I_{G_0} is convergent. One of the conditions for the covergence of I_{G_0} is $\omega(G_0) < 0$, or equivalently, $\omega(\gamma) + 2d - 8 < 0$. Let us consider the subgraph $\gamma \cup \{1, 2, 3\}$ next. For this subgraph $c(\gamma \cup \{1, 2, 3\}) = 2$. Condition (1) of theorem 2 says that the subgraph should satisfy $\omega(\gamma) + 3(d-2) - 2d < 0$ or, equivalently, $\omega(\gamma) < 6 - d$. The same restriction follows from eq. (3.12), since the degree of divergence of the subgraph $\gamma \cup \{1, 2, 3\} \subset G_0$ is $\omega = \omega(\gamma) + d - 6$. For $d \geq 2$ we have $6 - d \geq 8 - 2d$. Thus we are left with a single condition $\omega(\gamma) < 8 - 2d$. One may derive analogous relations for the subgraphs of γ.

Analysis of proposition 3. Let us denote the hashed block in eq. (3.16) by γ. The subgraph of G_g formed by γ and the lines $1,\ldots,n_1$ has $c=1$ and $\omega=\omega(\gamma)+n_1(d-2)$. The condition (1) of the theorem 2 reads as $\omega(\gamma)+n_1(d-2)-d<0$. Let us inspect the graph G_0 on the r.h.s. of eq. (3.16). The subgraph of G_0 formed by γ and the lines $1,\ldots,n_1$ has $\omega=\omega(\gamma)+n_1(d-2)-d$. Thus we have the same convergence condition that we found before.

4. Subtraction of divergences and counterterms

4.1 Subtraction of divergences

In section 3 we argued that the convergence theorem holds for the non-commutative scalar theories with non-derivative couplings. In what follows we assume that it holds also for theories with derivative couplings.

In this section we propose a non-commutative analog of Bogoliubov-Parasiuk's recursive subtraction formula and show that it leads to finite integrals. Our discussion will be parallel to one in the commutative QFT case and we refer the readers not familiar with the subject of BPHZ renormalization to the ref. [18] for a nice and elementary introduction.

For the reason given at the beginning of section 1.2, for a particular NQFT in d dimensions, we restrict our discussion to the graphs of class Ω_d. The class Ω_d consists of graphs whose *topologically non-trivial* subgraphs satisfy at least one of the conditions (1)–(2) of theorem 2. By definition, a subgraph $\gamma\subseteq G$ is topologically nontrivial (=non planar) if on Σ_g none of the closed paths in γ can be contracted to a point. Note that a topologically non-trivial graph is not necessarily homologically nontrivial.[14] Topologically nontrivial, but homologically trivial subgraphs have $c=0$ and so the condition (1) of theorem 2 for such graphs reads as $\omega<0$.

This means that if $G\in\Omega_d$, then only topologically trivial subgraphs of G are allowed to violate the conditions (1)–(2) of theorem 2. Our subtraction procedure renders the graphs from the class Ω_d finite. We will show that the recursion formula applied to the integrand $\mathcal{I}_G$ of a graph $G\in\Omega_d$ yields an expression which satisfies the conditions of the convergence theorem.

Let Σ_g be a particular genus g surface on which the graph G is drawn. There will be momenta flowing in the loops of the diagram, but as pointed out in section 3.2, only the momentum flow along the homologically non-trivial cycles contribute to the phase factor. Let p_{a_i}, p_{b_i} be the momenta flowing along the non-trivial cycles of Σ_g. Denoting by φ_G the phase factor for graph G, the general form of the integrand $\mathcal{I}_G$ of graph G in momentum space reads

$$\mathcal{I}_G\left(k,q,p_{a_i},p_{b_i}\right)=e^{i\varphi_G(k,p_{a_i},p_{b_i})}\mathcal{I}_G^{\theta=0}\left(k,q,p_{a_i},p_{b_i}\right), \tag{4.1}$$

[14] See footnote 7.

where $\mathcal{I}_G^{\theta=0}$ is the integrand for the corresponding commutative QFT, and k and q denote the external and the rest of independent loop momenta, respectively. If graph G is planar, then its phase factor $\varphi(k)$ depends only on the external momenta (see eq. (1.2)). Let us define $\mathcal{I}_G$ for a planar graph G to be

$$\mathcal{I}_G = \mathcal{I}_G^{\theta=0} \tag{4.2}$$

and $\mathcal{I}_G$ for a non-planar graph to be given by eq. (4.1).

For a topologically trivial graph G, let us denote by $R_G^{(0)}$ the renormalized integrand that leads to a finite integral. Let us denote by $\bar{R}_G^{(0)}$ the integrand with all subdivergences except the overall divergence of G subtracted. Let T_G be the operator which acts on $\bar{R}_G^{(0)}$ of a planar graph G as follows. $T_G \bar{R}_G^{(0)}$ is the Taylor expansion of $\bar{R}_G^{(0)}$ in the external momenta at the origin, up to the order $\omega(G)$ included. Let $\Re(G)$ be the set of all renormalization parts of the graph G, where by the renormalization part we mean any planar 1PI subgraph $\gamma \subset G$ except for G itself such that $\omega(\gamma) \geq 0$. When two subgraphs γ_1 and γ_2 have no common vertex nor line, we denote $\gamma_1 \cap \gamma_2 = \emptyset$. With these conventions, the recursive subtraction formula for a planar graph G reads

$$R_G^{(0)} = \begin{cases} \bar{R}_G^{(0)} & \text{if } \omega(G) < 0\,, \\ (1 - T_G)\, \bar{R}_G^{(0)} & \text{if } \omega(G) \geq 0\,, \end{cases}$$

$$\bar{R}_G^{(0)} = \mathcal{I}_G + \sum_{\substack{\{\gamma_1,\dots,\gamma_s\} \\ \gamma_j \in \Re(G),\ \gamma_i \cap \gamma_j = \emptyset}} \mathcal{I}_{G/\{\gamma_1,\dots,\gamma_s\}} \prod_{a=1}^{s} \left(-T_{\gamma_a} \bar{R}_{\gamma_a}^{(0)}\right). \tag{4.3}$$

Let us now consider a topologically non-trivial graph G. Let $\gamma_a \in \Re(G)$, $a = 1, \dots, s$, be a set of disjoint, $\gamma_i \cap \gamma_j = \emptyset$, renormalization parts. Since γ_i are topologically trivial we have

$$\mathcal{I}_{\gamma_i} = \mathcal{I}_{\gamma_i}^{\theta=0}\,.$$

The integrand of the reduced graph $G/\{\gamma_1, \dots, \gamma_s\}$ reads

$$\mathcal{I}_{G/\{\gamma_1,\dots,\gamma_s\}} = e^{i\varphi_G(k, p_{a_i}, p_{b_i})}\, \mathcal{I}_{G/\{\gamma_1,\dots,\gamma_s\}}^{\theta=0}\,. \tag{4.4}$$

The meaning of this equation is the following. When we shrink the renormalization parts $\gamma_1, \dots, \gamma_s$, the local structure of the graph G changes, but the global structure does not change because γ_i's are topologically trivial and disjoint. Thus the global flow of momentum remains unchanged, implying that the phase factor of the reduced graph $G/\{\gamma_1, \dots, \gamma_s\}$ is the same as that of G.

For a topologically non-trivial graph $G \in \Omega_d$, we define the renormalized integrand $R_G^{(1)}$ that leads to a finite integral as follows (see figure 13).

$$R_G^{(1)} = \mathcal{I}_G + \sum_{\substack{\{\gamma_1,\dots,\gamma_s\} \\ \gamma_j \in \Re(G),\ \gamma_i \cap \gamma_j = \emptyset}} \mathcal{I}_{G/\{\gamma_1,\dots,\gamma_s\}} \prod_{a=1}^{s} \left(-T_{\gamma_a} \bar{R}_{\gamma_a}^{(0)}\right), \tag{4.5}$$

where $\bar{R}^{(0)}_{\gamma_a}$ is given by eq. (4.3). Note that $R^{(1)}$ does not enter into the recursion, whereas $R^{(0)}$ does. In other words eq. (4.3) is recursive, whereas eq. (4.5) is non-recursive.

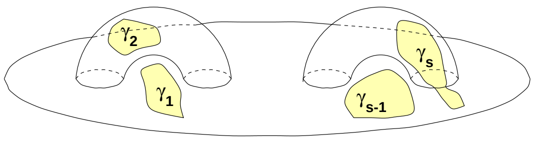


Figure 13: Topologically trivial disjoint renormalization parts.

For a general graph $G \in \Omega_d$ define

$$R_G = \begin{cases} R_G^{(0)} & \text{if } G \text{ is topologically trivial}\,, \\ R_G^{(1)} & \text{if } G \text{ is topologically nontrivial}\,. \end{cases} \tag{4.6}$$

Then the following theorem holds.

Theorem 3 *If the graph G belongs to the class Ω_d, then R_G leads to a finite integral.*

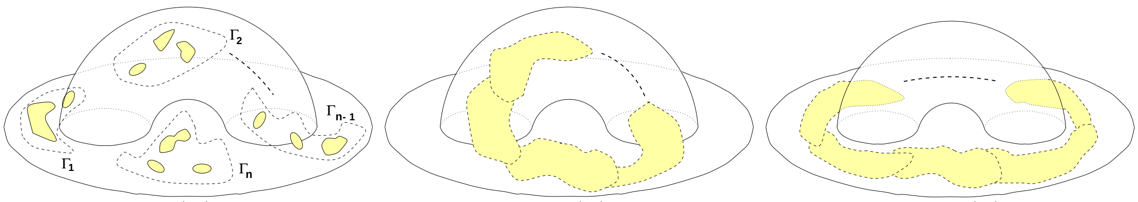
Proof. If G is planar, then $R_G = R_G^{(0)}$. It is known that the planar version of Bogoliubov-Parasuik's formula renders all divergent planar diagrams finite [23]. Thus we consider non-planar graph G. In this case $R_G = R_G^{(1)}$. Let us draw G on a surface Σ_g. The idea of the proof that R_G leads to a finite integral is simple. We just have to show that R_G satisfies the conditions of theorem 2. There are three potentially distinct cases to consider:

1. G has a disjoint set of topologically trivial proper 1PI subgraphs $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ such that each renormalization part $\gamma \in \Re(G)$ is contained in one of them (see figure 14a).

2. G has overlapping renormalization parts forming a subgraph γ_1 with $\omega(\gamma_1) \geq 0$, which wraps a homologically non-trivial cycle of Σ_g (see figure 14b).

3. G has overlapping renormalization parts forming a subgraph γ_2 with $\omega(\gamma_2) \geq 0$, which wraps a topologically nontrivial, but homologically trivial cycle of Σ_g (see figure 14c).

Let us analyze these cases:



(a) (b) (c)

Figure 14: Illustration for theorem 3.

Case 1. For each Γ_i in figure 14a, the subtracted integrand $R^{(0)}_{\Gamma_i}$ satisfies the condition $\omega < 0$. Let Γ be a subgraph of G which wraps a homologically non-trivial cycle of Σ_g. Let $\Gamma_1, \ldots, \Gamma_k \subset \Gamma$ be a disjoint set of topologically trivial 1PI subgraphs of Γ. Using the relation

$$\omega(\Gamma) = \omega\left(\frac{\Gamma}{\{\Gamma_1, \ldots, \Gamma_k\}}\right) + \sum_{a=1}^{k} \omega(\Gamma_a)$$

and the fact that $\omega(\Gamma) < d$, $\omega(R^{(0)}_{\Gamma_i}) < 0$, $i = 1, \ldots, k$, we find that

$$\omega(R_\Gamma) < d\,.$$

One can similarly show that all other subtracted subgraphs satisfy the conditions of theorem 2.

Cases 2 and 3. If G were not in the class Ω_d, for the subtraction of divergences of γ_1 and γ_2 one would have to introduce non-planar counterterms (geometrically it means that we pinch the handles of Σ_g). We do not know how to deal with this situation. But since G is assumed to be in the class Ω_d, we do not have to subtract the graphs γ_1 and γ_2 as a whole. Thus the global structure of the graph G remains unchanged as a result of the subtraction procedure. The argument given in the Case 1 then applies here as well. □

4.2 Generation of subtractions by counterterms

For a given scalar NQFT in d dimensions, we have seen how to renormalize an individual non-commutative Feynman graph G from the class Ω_d by applying the recursion formula to the integrand $\mathcal{I}_G$.

If scalar NQFT is not renormalizable in the commutative limit, then the class Ω_d is smaller than the class of all diagrams of the theory. In a commutative QFT it is possible to renormalize a non-renormalizable theory by including counterterms with an arbitrarily large number of powers of momentum and with an arbitrarily large number of external lines. Our subtraction procedure works only for the graphs from the class Ω_d. Thus if NQFT is not renormalizable in the commutative limit, then it is not possible to renormalize an arbitrary graph by the introduction of counterterms in the action of the form

$$\operatorname{tr} L_{ct}\left(\Phi(\hat{x}), \hat{\partial}\Phi(\hat{x}), \hat{\partial}\hat{\partial}\Phi(\hat{x}), \ldots\right),$$

where $\hat{\partial}_\mu \equiv \theta^{-1}_{\mu\nu}[\hat{x}^\nu, \cdot]$ is the non-commutative analog of the derivative.

Unfortunately, even for the scalar field theories which are renormalizable in the commutative limit $\theta = 0$ (e.g. $\phi \star \phi \star \phi$ theory in six dimensions) the class Ω_d is

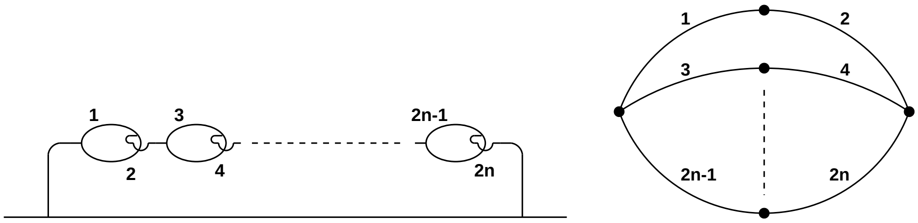


Figure 15: (a) Minwalla-Raamsdonk-Seiberg's counter-example. (b) Representation of P_2.

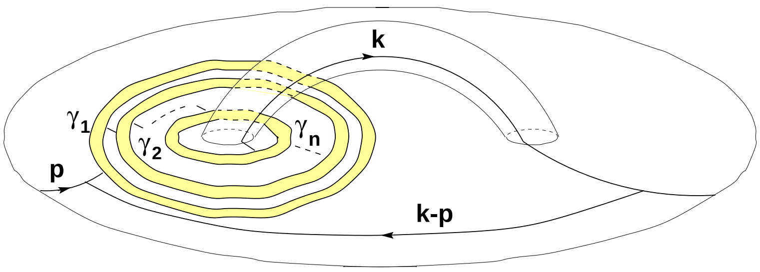


Figure 16: A general diagram not included in class Ω_6.

smaller than the class of all diagrams of the theory.[15] In ref. [22], the diagram in figure 15a is shown to be divergent in six dimensions for $n \geq 3$. An easy way to see this is to note that P_2 for this graph is equal to P_0 for the graph in figure 15b (see section 2.2 for the definition of P_0 and P_2) The graph in figure 15b is divergent in six dimensions for $n \geq 3$.

The other way to see the divergence of the graph in figure 15 in six dimensions is to note that the disjoint subgraph γ formed by the lines $1, 2, \ldots, 2n-1, 2n$ has $\omega(\gamma) = 6n - 4n = 2n$ and $c(\gamma) = 1$. It means that the condition (1) of theorem 2 is violated if $n \geq 3$: $\omega(\gamma) - 6 \geq 0$.

In general the graphs of the type shown in figure 16 are not in the class Ω_6 for the $\phi\star\phi\star\phi$ theory. The reason is the following. Each of the subgraphs γ_i in figure 16 has two-external lines and thus $\omega(\gamma_i) = 2$. But $c(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n) = 1$.

Let us show, on the example of $\phi\star\phi\star\phi$ theory in six dimensions, that the recursive subtraction procedure of section 4.1 is equivalent to the counterterm approach. Although the subtraction procedure of section 4.1 is incapable of removing all divergences of the theory, the analysis given in this section is useful for the discussion about Wess-Zumino model given in section 5. The following discussion is completely parallel to the one given in ref. [24] for the commutative QFT. Let us decompose the lagrangian as follows:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_b + \mathcal{L}_{ct}\,. \tag{4.7}$$

[15]This was pointed out to us by Shiraz Minwalla and Mark Van Raamsdonk. The counter-example is given in figure 15. In the original version of this paper, we stated that if NQFT is renormalizable in the commutative limit, then the class Ω_d is *equal* to the class of all diagrams of the theory. This wrong statement led us to conclude that $\phi \star \phi \star \phi$ theory in six dimensions is renormalizable.

JHEP05(2000)037

Here $\mathcal{L}_0$ is the free lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\,(\partial\phi)^2 + \frac{m^2}{2}\,\phi^2 \tag{4.8}$$

with m being the renormalized mass. The rest of the lagrangian, $\mathcal{L}_I = \mathcal{L}_b + \mathcal{L}_{ct}$,is the interaction, and consists of two terms. The first, which we will call the basic interaction, is

$$\mathcal{L}_b = \left(\frac{g}{3}\right)(\phi\star\phi\star\phi)\,. \tag{4.9}$$

The second term is the counterterm lagrangian and it is defined as follows. Let $C_2(G,k_1,k_2)$ be the overall counterterm for the planar 1PI graph G with two external lines. Let $C_3(G,k_1,k_2,k_3)$ be the overall counterterm for the planar 1PI graph G with three external lines. Note that C_2 and C_3 do not contain the phase factors associated with the external momenta. Then the counterterm action S_{ct} reads

$$\begin{aligned} S_{ct} = & \sum_{\text{2-point } G} \frac{1}{2}\int d^6k_1\, d^6k_2\, d^6k_3\, \phi(k_1)C_2(G,k_1,k_2)\phi(k_2) + \\ & + \sum_{\text{3-point } G} \frac{1}{3}\int d^6k_1\, d^6k_2\, d^6k_3\, \phi(k_1)\phi(k_2)\phi(k_3)C_3(G,k_1,k_2,k_3)\times \\ & \times \exp\Big(i\,\theta_{\mu\nu}\,(k_1^\mu\,k_2^\nu + k_1^\mu\,k_3^\nu + k_2^\mu\,k_3^\nu)\Big). \end{aligned} \tag{4.10}$$

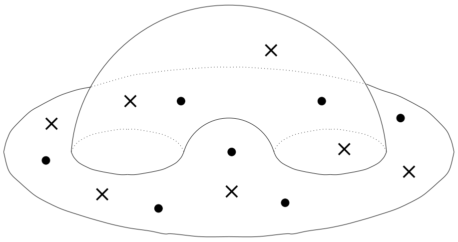
Thus the counterterm lagrangian is

$$\mathcal{L}_{ct} = \frac{\delta Z(\partial\phi)^2}{2} + \frac{\delta m^2\phi^2}{2} + \frac{\delta g(\phi\star\phi\star\phi)}{3} \tag{4.11}$$

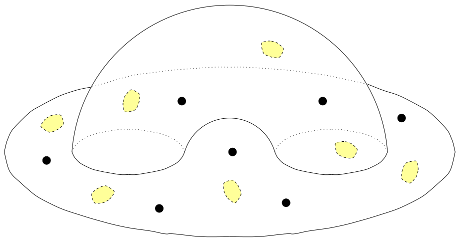
with

$$\begin{aligned} \delta Z &= \sum_{\text{2-point } G} \big[\text{Coefficient of } -p^2 \text{ in } C_2(G)\big]\,, \\ \delta m^2 &= \sum_{\text{2-point } G} \big[\text{Coefficient of } p^0 \text{ in } C_2(G)\big]\,, \\ \delta g &= \sum_{\text{3-point } G} C_3(G)\,. \end{aligned} \tag{4.12}$$

Consider the full N-point Green's function G_N at order g^L in the NQFT with the lagrangian given by eq. (4.7). The term of order g^L in the perturbation expansion of G_N has vertices generated by the different terms in $\mathcal{L}_b+\mathcal{L}_{ct}$. There will be graphs with all of their vertices being the basic interaction $\mathcal{L}_b$. The other graphs will contain one or more of the counterterm vertices generated by $\mathcal{L}_{ct}$ eq. (4.11). A generic diagram which contains counterterm vertices looks like the one shown in figure 17a. If we replace each counterterm vertex in the graph in figure 17a by the sum over overall-divergent 1PI graphs as in eq. (4.12), then each term in the resulting multiple sum



(a)



(b)

Figure 17: The counterterm and basic graphs. $\times$ and $\bullet$ stand for the counterterm and basic vertices, respectively.

corresponds to a unique basic graph as in figure 17*b*. On the other hand, according to the subtraction formula eq. (4.5), for a graph G of genus g we subtract all possible disjoint unions of divergent topologically trivial (planar) 1PI subgraphs. The analysis is completely parallel to the one in commutative case [24], with a simplification due to the combinatorics in the non-commutative case. The point is that all ribbon graphs come with the combinatorial factor 1. Thus the recursive subtraction procedure of section 4.1 is equivalent to the counterterm approach.

5. Conclusions and discussions

We proved the convergence of some classes of diagrams in massive scalar quantum field theories on non-commutative $\mathbb{R}^d$ and formulated a general convergence theorem for the non-commutative Feynman graphs. Although we did not prove the convergence theorem in its general form, we made it very plausible by demonstrating its universal character. We should also mention that we analyzed numerous other examples not discussed in this paper and found that they are in complete agreement with the statements of the general convergence theorem.

We proposed a recursive subtraction formula for divergent Feynman graphs and showed that for the graphs in class Ω_d it leads to finite integrals. For a generic scalar non-commutative quantum field theory on $\mathbb{R}^d$, the class Ω_d is smaller than the class of all diagrams in the theory. This leaves open the question of perturbative renormalizability of non-commutative field theories. As explained in section 4.2, the problematic graphs (the graphs that are not in the class Ω_d) are of the type shown in figure 16. All the rings $\gamma_1, \ldots, \gamma_n$ wrap a single cycle, but each ring has $\omega > 0$. For a large-enough number of rings, the subgraph formed by their disjoint union will not satisfy the condition (1) of theorem 2, i.e. the graph will diverge. A natural way to avoid the violation of condition (1) in theorem 2 by the accumulation of positive ω's is to enforce the condition $\omega \leq 0$ for the subgraphs. This situation is realized in supersymmetric theories.[16] As an example consider a non-commutative version of

[16]The non-supersymmetric $\phi\star\phi\star\phi$ theory in four dimensions also has only logarithmic divergences,

supersymmetric Wess-Zumino model in four dimensions:

$$S[\Phi] = \int d^4x\, d^2\theta\, d^2\bar{\theta}^2\, \Phi^+ \star \Phi + \left\{ \int d^4x\, d^2\theta \left[m\Phi^2 + g\ \Phi \star \Phi \star \Phi \right] + \text{h.c.} \right\}.$$

It is well known that commutative Wess-Zumino model has only logarithmic divergences. Thus it is plausible that the non-commutative Wess-Zumino model is renormalizable.

Acknowledgments

We are grateful to Y.M. Makeenko, H. Nastase, M. Roček and G. Sterman for useful discussions and suggestions. We would also like to thank S. Minwalla and M.V. Raamsdonk for pointing out a mistake regarding the class Ω_d in the original version of this paper. This work was supported in part by NSF grant PHY-9722101.

A. Parametric integral representation

Using the integral representations for the propagators and the δ^d function in eq. (2.11)[17] one finds

$$\begin{aligned} &\int_0^\infty \prod_{l=1}^{I} d\alpha_l \int \prod_{v=1}^{V} d^d y_v e^{-\sum_l \alpha_l m_l^2} \int \prod_l d^d\, k_l \\ &\qquad \exp\left[-\sum_l a_l k_l^2 - i \sum_v y_v \cdot \left(P_v - \sum_l \epsilon_{vl}\, k_l \right) + \right. \\ &\qquad\qquad \left. + i \left(\sum_{m,n} I_{mn}\, \theta_{\mu\nu}\, k_m^\mu k_n^\nu + \sum_{m,v} J_{mv}\, \theta_{\mu\nu}\, k_m^\mu\, P_v^\nu \right) \right]. \end{aligned} \tag{A.1}$$

Integration over the momenta k in eq. (A.1) gives

$$\begin{aligned} I_G(P) = \pi^{Id/2} &\int_0^\infty \prod_l d\alpha_l\, e^{-\sum_l \alpha_l m_l^2} \int \prod_v d^d y_v\, e^{-i\sum_v y_v \cdot P_v} (\det \mathcal{A})^{-1/2} \times \\ &\times \exp\left\{ -\frac{1}{4} \left[(J\eta)_m^\mu + (y\epsilon)_m^\mu \right] \left(\mathcal{A}^{-1} \right)_{mn}^{\mu\nu} \left[(J\eta)_n^\nu + (y\epsilon)_n^\nu \right] \right\}. \end{aligned} \tag{A.2}$$

Making the following change of integration variables:

$$\begin{aligned} y_1 &= z_1 + z_V\,, \\ y_2 &= z_2 + z_V\,, \\ &\vdots \\ y_{V-1} &= z_{V-1} + z_V\,, \\ y_v &= z_V \end{aligned} \tag{A.3}$$

but it is a trivial theory.

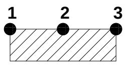
[17] See ref. [18].

the jacobian of which is one, and using the fact that $\sum_v \epsilon_{vl} = 0$, one finds

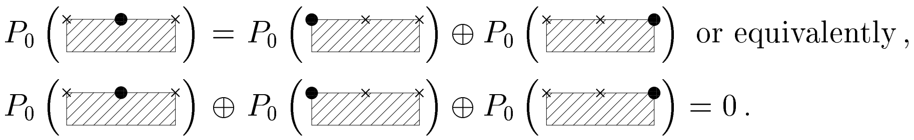
$$
\begin{aligned}
I_G(P) = \pi^{Id/2}(2\pi)^d\,\delta^{(d)}\left(\sum_v P_v\right)\int_0^\infty \prod_{l=1}^{I} d\alpha_l\, e^{-\sum_l \alpha_l m_l^2}(\det\mathcal{A})^{-1/2}\times \\
\times \int \prod_{v=1}^{V-1} d^d z_v\, e^{-i\sum_{v=1}^{V-1} z_v\cdot P_v}\times \\
\times\exp\left\{-\frac{1}{4}\Big[(J\eta)_m^\mu+(z\epsilon_V)_m^\mu\Big]\left(\mathcal{A}^{-1}\right)_{mn}^{\mu\nu}\Big[(J\eta)_n^\nu+(z\epsilon_V)_n^\nu\Big]\right\}, \qquad \text{(A.4)}
\end{aligned}
$$

where $(z\epsilon_V)_m^\mu \equiv \sum_{v=1}^{V-1} z_v^\mu \epsilon_{vm}$. Performing the z-integrals, one finds eq. (2.12).

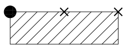
B. Proof of the lemma

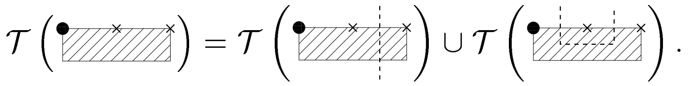
For simplicity we prove the lemma for the $n = 3$ case. Generalization to the case of arbitrary n is straightforward. Let us consider a graph  with three of its vertices labeled. We will denote by crosses vertices that are identified.

For $n = 3$ eq. (3.11) reads



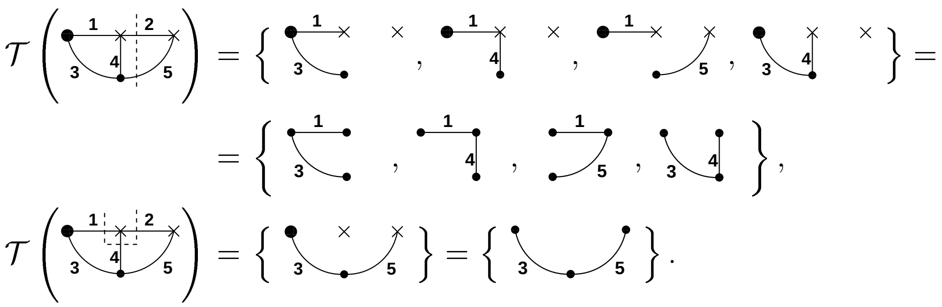
(B.1)

The set of trees of the graph  can be written as the union of two sets: the set of trees of that directly link vertices 1 and 2, and the set of trees of that directly link vertices 1 and 3

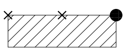
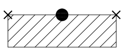


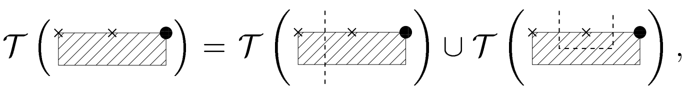
(B.2)

Example 11

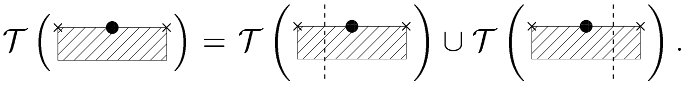


(B.3)

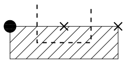
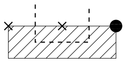
Similarly, we can split the trees of  and  as follows:



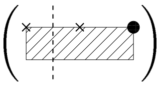
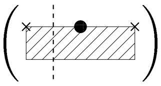
(B.4)



(B.5)

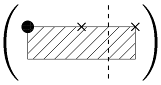
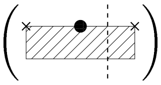
We notice that the chords associated with  are identical to the chords associated with  and therefore cancel in $\oplus$ sum. Using the relations

$$\mathcal{T}^* \left(\quad \right) = \mathcal{T}^* \left(\quad \right) \tag{B.6}$$

and

$$\mathcal{T}^* \left(\quad \right) = \mathcal{T}^* \left(\quad \right), \tag{B.7}$$

where $\mathcal{T}^*$ denotes the chord set, it is easy to verify that eq. (B.1) holds.

References

[1] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[2] A. Connes, *Essay on physics and non-commutative geometry* in *The interface of mathematics and particle physics*, D.G. Quillen et al. eds., Clarendon press, Oxford 1990.

[3] A. Connes and J. Lott, *Particle models and noncommutative geometry*, *Nucl. Phys.* **18B** *(Proc. Suppl.)* (1991) 29.

[4] S. Doplicher, K. Fredenhagen and J.E. Roberts, *Space-time quantization induced by classical gravity*, *Phys. Lett.* **B 331** (1994) 39;
The quantum structure of space-time at the Planck scale and quantum fields, *Comm. Math. Phys.* **172** (1995) 187.

[5] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53.

[6] J.C. Varilly and J.M. Gracia-Bondia, *On the ultraviolet behaviour of quantum fields over noncommutative manifolds*, *Int. J. Mod. Phys.* **A 14** (1999) 1305 [hep-th/9804001];
M. Chaichian, A. Demichev and P. Presnajder, *Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences*, *Nucl. Phys.* **B 567** (2000) 360 [hep-th/9812180].

[7] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[8] M.R. Douglas, *Two lectures on D-geometry and noncommutative geometry*, hep-th/9901146.

[9] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[10] C.P. Martin and D. Sanchez-Ruiz, *The one-loop uv divergent structure of* U(1) *Yang-Mills theory on noncommutative* R^4, *Phys. Rev. Lett.* **83** (1999) 476 [hep-th/9903077];
M.M. Sheikh-Jabbari, *Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus*, *J. High Energy Phys.* **06** (1999) 015 [hep-th/9903107];

T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, hep-th/9903187.

[11] M. Li, *Strings from IIB matrices*, *Nucl. Phys.* **B 499** (1997) 149 [hep-th/9612222];
H. Aoki N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys.* **B 565** (2000) 176 [hep-th/9908141];
N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 573 [hep-th/9910004];
I. Bars and D. Minic, *Non-commutative geometry on a discrete periodic lattice and gauge theory*, hep-th/9910091;
J. Ambjorn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Finite-N matrix models of noncommutative gauge theory*, *J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041].

[12] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, hep-th/9908056.

[13] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[14] J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134].

[15] I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative* $\mathbb{R}^d$*, 2. Yang-Mills*, work in progress.

[16] G. Springer, *Introduction to Riemann surfaces*, Chelsea Pub. Co., New York 1981.

[17] T. Eguchi and R. Nakayama, *Simplification of quenching procedure for large-N spin models*, *Phys. Lett.* **B 122** (1983) 59.

[18] C. Itzykson and J.-B. Zuber, *Quantum field theory* McGraw-Hill, New York 1980.

[19] I. Chepelev and R. Roiban, *Convergence theorem for noncommutative Feynman graphs*, work in progress.

[20] S. Weinberg, *High-energy behaviour in quantum field theory*, *Phys. Rev.* **118** (1960) 838.

[21] W.E. Caswell and A.D. Kennedy, *The asymptotic behavior of Feynman integrals*, *Phys. Rev.* **D 28** (1983) 3073.

[22] S. Minwalla, M.V. Raamsdonk and N. Seiberg, private communication.

[23] V. Rivasseau, *Construction and borel summability of planar four-dimensional euclidean field theory*, *Comm. Math. Phys.* **95** (1984) 445.

[24] J. Collins, *Renormalization*, Cambridge University Press, Cambridge 1984.

JHEP05(2000)037